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ON THE LATERAL BUCKLING OF UNIFORM SLENDER CANTILEVER BEAMS

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Abstract—The general lateral buckling equation is developed for a uniform, slender cantilever beam with a load applied at the shear center of the end cross section. This equation is then specialized to include only the first order effect of the principal bending curvature revealing errors in previous first order analyses. These errors resulted from a failure to properly distinguish between the geometric and elastic angles of twist. The correct specialized equation is actually simpler than previously published equations and results in a buckling load formula noticeably different from formulas based on these earlier equations. This present buckling load formula is shown to compare favorably with a numerical solution of the general equation.

NOTATION

- A bending stiffness in plane of greatest flexural rigidity
- b/h ratio of cross section dimensions, $b/h < 1$
- B bending stiffness in plane of least flexural rigidity
- C torsion stiffness constant
- $J_{-1/4}$ Bessell function of first kind, order $-\frac{1}{4}$
- l length of beam
- N_1 shear force in plane of least flexural rigidity
- N_2 shear force in plane of greatest flexural rigidity
- P tip load applied in plane of greatest flexural rigidity
- s arc length of deformed beam shear center axis
- T tension
- y lateral deflection measured from beam tip
- α Reissner parameter $\beta^2(1-B/A)$
- β nondimensional buckling load $\equiv Pl^2/\sqrt{BC}$
- β_0 classical nondimensional buckling load $= 4{\cdot}0126$
- γ Reissner parameter $1-C/A$
- ζ slope in plane of greatest flexural rigidity
- κ curvature in plane of greatest flexural rigidity
- λ curvature in plane of least flexural rigidity
- ν Poisson's ratio
- σ dimensionless arc length
- τ elastic rate of twist
- ϕ elastic angle of twist
- $\hat{\phi}$ geometric angle of twist
- $(\ )_0$ $(\ )$ evaluated in the pre-buckled configuration (also, variables expressed in asymptotic expansion form may be subscripted $(\ )_0$, $(\ )_1$, $(\ )_2$, etc.).

1. INTRODUCTION

The classical theory for the lateral buckling of deep cantilever beams was first presented by Michell[1]. In this theory, bending curvature in the plane of greatest flexural rigidity prior to buckling is completely neglected. Prandtl[2] independently developed this same theory and also generalized it to include the first order effect of the principal bending curvature deriving an approximate buckling load formula. Reissner[3], unaware of Prandtl's generalized theory, developed a separate theory for the first order effect of the principal bending curvature. Reissner developed a series solution for the buckling load, and Federhofer[4] used Reissner's theory to generate numerical values of the buckling load.† In addition, these numerical results are compared in [4] to results obtained from Prandtl's formula[2] and found to be approximately equal for deep cross sections. However, a closer examination of the analyses of [2, 3] indicates that the two theories are not equivalent to first order. Of historical interest is the fact that Timoshenko and Gere[5] were not aware of Prandtl's generalized theory, in spite of Federhofer's

†Timoshenko and Gere [5] mention the work of Dinnik[6] in connection with the effect of the principal bending curvature although Dinnik did not actually treat this particular effect.

work. The same can be said of Goodier[7], who gave Reissner credit for results that appear in [1] and [2].

In this paper an improved derivation of the lateral buckling equation is given. First, the general equation is developed including the complete effect of the principal bending curvature prior to buckling. This general equation is then specialized to include only the first order effect of the principal bending curvature. Two alternate forms of the specialized equation are also derived. Comparison of these derivations with the derivations of Prandtl and Reissner reveals that the previous analyses[2, 3] are in error. The correct version of the specialized equation, actually simpler than previously published equations, is used to generate a buckling load formula. This formula is compared with similar formulas derived from the equations of [1–3]. In addition, numerical results obtained using the present formula are compared with a numerical solution of the general equation. Finally, these results are compared with those from the formulas of [1, 2] and with a series solution from [3].

2. DERIVATION OF BUCKLING EQUATIONS

2.1 *The general τ equation*

For uniform slender beams, the warping rigidity and the tension-torsion coupling may be neglected[8]. Thus, the Kirchoff equilibrium equations for an initially straight beam loaded only at the ends as given by Love[9] are applicable for deriving the buckling equations.

$$\left.\begin{aligned} &\frac{dN_1}{ds} - N_2\tau + T\lambda = 0 \\ &\frac{dN_2}{ds} - T\kappa + N_1\tau = 0 \\ &\frac{dT}{ds} - N_1\lambda + N_2\kappa = 0 \\ &A\frac{d\kappa}{ds} - (B - C)\lambda\tau = N_2 \\ &B\frac{d\lambda}{ds} + (A - C)\kappa\tau = -N_1 \\ &C\frac{d\tau}{ds} - (A - B)\kappa\lambda = 0 \end{aligned}\right\}. \tag{2.1}$$

The deformed beam is shown in Fig. 1. The curvatures are κ and λ in and out of the plane of the major axis, respectively. The rate of twist τ is given by $\tau \equiv (d\phi/ds)$ where ϕ is the usual elastic torsional deformation; ϕ is positive for counter-clockwise rotations of the cross section when the beam is viewed from tip to root. The principal bending stiffnesses are A and B, where $A > B$; and C is the torsional rigidity. N_1 and N_2 are shear forces and T is the tension. The arc length s is measured from the tip of the deformed beam. We assume first that the beam is loaded in the plane of greatest flexural rigidity with the buckling load P and that buckling has not occurred. Thus, $N_{1_0} = \tau_0 = \lambda_0 = 0$ prior to buckling and we shall then look at small perturbations with respect to this position. Equations (2.1) for the unbuckled configuration yield

$$\left.\begin{aligned} &\frac{dN_{2_0}}{ds} - T_0\kappa_0 = 0 \\ &\frac{dT_0}{ds} + N_{2_0}\kappa_0 = 0 \\ &A\frac{d\kappa_0}{ds} = N_{2_0} \end{aligned}\right\} \tag{2.2}$$

From summing forces on an end beam element (Fig. 2) we have

$$\begin{aligned} T_0 &= P \sin \zeta_0 \\ N_{2_0} &= P \cos \zeta_0 \end{aligned} \tag{2.3}$$

where ζ_0 is the slope, positive down, of the unbuckled beam at s. Equations (2.2) are satisfied if

$$\left.\begin{aligned} \kappa_0 &= -\frac{\mathrm{d}\zeta_0}{\mathrm{d}s} \\ \frac{\mathrm{d}^2\zeta_0}{\mathrm{d}s^2} + \frac{P}{A}\cos\zeta_0 &= 0 \\ \zeta_0(l) = \frac{\mathrm{d}\zeta_0}{\mathrm{d}s}(0) &= 0 \end{aligned}\right\} \tag{2.4}$$

The exact solution to eqns (2.4) may be written in terms of elliptic functions. The linearized perturbation equations for N_1, τ, and λ are

$$\left.\begin{aligned} \frac{1}{P}\frac{\mathrm{d}N_1}{\mathrm{d}s} &= \tau\cos\zeta_0 - \lambda\sin\zeta_0 \\ B\frac{\mathrm{d}\lambda}{\mathrm{d}s} + (A-C)\kappa_0\tau + N_1 &= 0 \\ C\frac{\mathrm{d}\tau}{\mathrm{d}s} - (A-B)\kappa_0\lambda &= 0 \end{aligned}\right\} \tag{2.5}$$

Thus, the linear buckling equation (for τ) may be written by using (2.4) and (2.5)

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(\frac{1}{\kappa_0}\frac{\mathrm{d}\tau}{\mathrm{d}s}\right) + \frac{(A-B)(A-C)}{BC}\frac{\mathrm{d}}{\mathrm{d}s}(\kappa_0\tau) + \frac{A(A-B)}{BC}\frac{\mathrm{d}\kappa_0}{\mathrm{d}s}\tau - \frac{A}{B}\frac{1}{\kappa_0}\frac{\mathrm{d}\kappa_0}{\mathrm{d}s}\frac{\mathrm{d}\tau}{\mathrm{d}s}\tan\zeta_0 = 0. \tag{2.6}$$

The boundary conditions follow from $\lambda(0) = \kappa_0(0) = \tau(0) = 0$ at the free end, $\lambda(l) = 0$ at the root, and the third of eqns (2.5).

$$\tau(0) = \frac{\mathrm{d}\tau}{\mathrm{d}s}(l) = \frac{\mathrm{d}^2\tau}{\mathrm{d}s^2}(0) = 0.$$

Equations (2.4) and (2.6) comprise a general theory for the lateral buckling of uniform, slender cantilever beams.

2.2 *The specialized τ equation*

As an approximation to the general theory we consider letting ζ_0^2 be neglected with respect to unity. Equations (2.4) become

$$\frac{\mathrm{d}^2\zeta_0}{\mathrm{d}s^2} = -\frac{P}{A}. \tag{2.7}$$

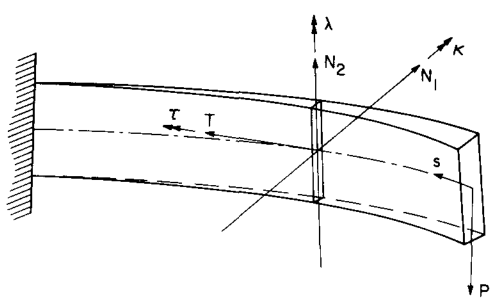

Fig. 1.

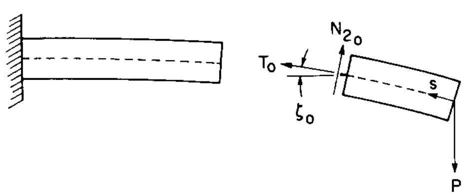

Fig. 2.

Therefore,

$$\frac{d\zeta_0}{ds} = -\frac{Ps}{A} = -\kappa_0$$

$$\zeta_0 = \frac{P}{2A}(l^2 - s^2). \tag{2.8}$$

Equation (2.6) then becomes

$$s\frac{d^2}{ds^2}\left(\frac{1}{s}\frac{d\tau}{ds}\right) + \frac{P^2}{BC}\left(1-\frac{B}{A}\right)\left(1-\frac{C}{A}\right)s\frac{d}{ds}(s\tau) + \frac{P^2}{BC}\left(1-\frac{B}{A}\right)s\tau - \frac{P^2}{BC}\frac{C}{2A}(l^2-s^2)\frac{d\tau}{ds} = 0. \tag{2.9}$$

This equation can be further simplified by consistently applying the ordering assumption $\zeta_0^2 \ll 1$. From eqn (2.8), we see that ζ_0 is maximum at the tip implying that

$$\zeta_0^2 \leq \frac{P^2 l^4}{4BC}\frac{BC}{A^2} \ll 1. \tag{2.10}$$

But since P is the buckling load, we know from [5] that $P^2l^4/BC > 16$. Therefore, the assumption $\zeta_0^2 \ll 1$ also implies $BC/A^2 \ll 1$. Neglecting BC/A^2 with respect to unity in eqn (2.9) results in the following first integral

$$s^2\frac{d}{ds}\left(\frac{1}{s^2}\frac{d\tau}{ds}\right) + \frac{P^2}{BC}\left[\left(1-\frac{B}{A}-\frac{C}{2A}\right)s^2 - \frac{C}{2A}l^2\right]\tau = 0 \tag{2.11}$$

where we have used the following identity

$$s\frac{d^2}{ds^2}\left(\frac{1}{s}\frac{d\tau}{ds}\right) \equiv \frac{d}{ds}\left[s^2\frac{d}{ds}\left(\frac{1}{s^2}\frac{d\tau}{ds}\right)\right]. \tag{2.12}$$

Equation (2.11) is the specialized τ equation. A more restrictive approximation $C/A \ll 1$, yields another first integral in ϕ where $\tau = (d\phi/ds)$

$$\frac{d^2\phi}{ds^2} + \frac{P^2}{BC}\left(1-\frac{B}{A}\right)s^2\phi = 0$$

$$\phi(l) = \frac{d\phi}{ds}(0) = 0. \tag{2.13}$$

This is the classical lateral buckling equation for a deep cantilever beam[1]. Considering, however, that B and C are often the same order of magnitude, the B/A term is usually neglected along with the C/A term in the classical theory[2, 5]. This is equivalent to neglecting all effects of principal bending curvature prior to buckling and yields a reasonable approximation for the buckling load of deep beams.

2.3 *The specialized N_1 equation*

An equation for N_1 may also be obtained from eqns (2.4) and (2.5). The N_1 equation has a simpler form than the specialized τ eqn (2.11) and provides a basis of comparison with the work of Reissner[3]. Rewriting eqns (2.5) for $\zeta_0^2 \ll 1$ leads to

$$\frac{1}{P}\frac{dN_1}{ds} = \tau - \zeta_0\lambda \tag{2.14}$$

$$B\frac{d\lambda}{ds} + (A-C)\kappa_0\tau + N_1 = 0 \tag{2.15}$$

$$C\frac{d\tau}{ds} - (A-B)\kappa_0\lambda = 0 \tag{2.16}$$

where κ_0 and ζ_0 are given by eqn (2.8). A first integral of eqn (2.15) may now be obtained by adding ζ_0 times eqn (2.16) to eqn (2.15). (The motivation for this combination resulted from a separate derivation of the N_1 equation via the variational approach.)

$$B\frac{d\lambda}{ds}+(A-C)\kappa_0\tau+C\zeta_0\frac{d\tau}{ds}-(A-B)\zeta_0\kappa_0\lambda+N_1=0. \tag{2.17}$$

Equation (2.17) can be simplified by the use of eqns (2.8) and (2.14).

$$B\left(\frac{d\lambda}{ds}+\lambda\kappa_0\zeta_0\right)+C\frac{d}{ds}(\zeta_0\tau)+\frac{d}{ds}(sN_1)=0. \tag{2.18}$$

It can be shown that max $|(d\lambda/ds)|>|(\lambda/l)|$ for all s. Therefore, to be consistent with the ordering scheme $\zeta_0{}^2\ll 1$, the $B\lambda\zeta_0\kappa_0$ term must be neglected, yielding a first integral of the equation:

$$B\lambda+C\zeta_0\tau+sN_1=0 \tag{2.19}$$

where we have used $\lambda(0)=\tau(0)=0$. Now τ and λ may be eliminated from eqn (2.14) by use of (2.16) and (2.19). The N_1 equation may then be obtained by using the inequality max $|(d^2N_1/ds^2)|>(1/l)|(dN_1/ds)|$, consistently neglecting terms of order $\zeta_0{}^2$ with respect to unity, and applying the ζ_0, κ_0 formulas of equation (2.8)

$$\frac{d^2N_1}{ds^2}+\frac{P^2}{BC}\left[s^2\left(1-\frac{B}{A}+\frac{3C}{2A}\right)-l^2\frac{C}{2A}\right]N_1=0 \tag{2.20}$$

with the corresponding boundary conditions

$$N_1(l)=\frac{dN_1}{ds}(0)=0.$$

Equation (2.20) is the specialized N_1 equation, the most concise form of the lateral buckling equation for the specialized theory.

2.4 *The specialized y equation*

To facilitate comparison with Prandtl's derivation[2], we also write the buckling equation for the lateral deflection y measured from the tip ($y(0)=0$). Thus according to [8], $(d^2y/ds^2)=\lambda+\kappa_0\phi$. We now substitute eqn (2.19) into eqn (2.16) for $B\lambda$. (The motivation for this substitution resulted from a separate derivation of the buckling equation via the Newtonian method.)

$$C\frac{d^2\phi}{ds^2}=A\kappa_0\lambda+\kappa_0\left(sN_1+C\zeta_0\frac{d\phi}{ds}\right). \tag{2.21}$$

We observe that according to eqn (2.14) and the assumption that $\zeta_0{}^2\ll 1$

$$C\frac{d^2\phi}{ds^2}=Ps(\lambda+\kappa_0\phi)=Ps\frac{d^2y}{ds^2}. \tag{2.22}$$

Thus,

$$C\frac{d\phi}{ds}=P\left(s\frac{dy}{ds}-y\right)=Ps^2\frac{d}{ds}\left(\frac{y}{s}\right). \tag{2.23}$$

Another equation for ϕ and y may be written from eqns (2.19) and (2.23)

$$\frac{d^2y}{ds^2}-\frac{Ps\phi}{A}+\frac{P\zeta_0}{B}\left(s\frac{dy}{ds}-y\right)+\frac{sN_1}{B}=0 \tag{2.24}$$

where, from eqn (2.14) and $\zeta_0^2 \ll 1$

$$N_1 = P\left(\phi + \int_s^l \zeta_0 \frac{d^2y}{ds^2} ds\right). \tag{2.25}$$

When we divide eqn (2.24) by s, differentiate with respect to s, and substitute for $(d\phi/ds)$ from eqn (2.23), we obtain a single third order equation for y

$$\frac{d}{ds}\left(\frac{1}{s}\frac{d^2y}{ds^2}\right) + \frac{P^2}{BC}\left[\left(1 - \frac{B}{A} - \frac{C}{2A}\right)s^2 - \frac{C}{2A}l^2\right]\frac{d}{ds}\left(\frac{y}{s}\right) = 0 \tag{2.26}$$

with the boundary conditions

$$y(0) = \frac{dy}{ds}(l) = \frac{d^2y}{ds^2}(0) = 0.$$

Equations (2.22) and (2.23) may be substituted into eqn (2.26) to yield an equation in τ identical to eqn (2.11). All three specialized buckling eqns (2.11), (2.20) and (2.26) will yield the same first order buckling load formula.

3. COMPARISON WITH PREVIOUS BUCKLING ANALYSES

3.1 *Prandtl's analysis*

Prandtl[2] derived a single lateral buckling equation in terms of the lateral deflection y generalized to include the effect of principal bending curvature. The analysis was based on the Newtonian method with the assumption that $\zeta_0^2 \ll 1$. Equation (2.26) may be rewritten for comparison with Prandtl's final equation.

$$s\frac{d^3y}{ds^3} - \frac{d^2y}{ds^2} + \frac{P^2}{BC}\left\{\left[\left(1 - \frac{B}{A}\right)s^2 - \frac{C}{2A}l^2\right]\left(s\frac{dy}{ds} - y\right) - \frac{C}{2A}s^2\left(s\underline{\frac{dy}{ds}} - y\right)\right\} = 0. \tag{3.1}$$

Equation (3.1) is identical to eqn (35) of [2] with the exception that the underlined term has the opposite sign in [2]. An examination of Prandtl's analysis reveals that the discrepancy results from the evaluation of N_1 in eqn (2.24). In the text of [2], directly following eqn (33), Prandtl's equilibrium equation is expressed in a form similar to eqn (2.24) with the exception that N_1 is given by

$$N_1 = P\left(\phi - \zeta_0\frac{dy}{ds}\right) \quad \text{(Prandtl)} \tag{3.2}$$

which is incorrect. The correct expression for N_1 is given by eqn (2.25), the first integral of Kirchoff's equation, (2.14). The presence of this error introduces the sign reversal into eqn (3.1) and precludes the possibility of writing eqn (3.1) in the more compact form, eqn (2.26).

3.2 *Reissner's analysis*

Reissner[3] derived a single lateral buckling equation in terms of the lateral shear N_1 for the same generalized problem treated by Prandtl. Although the work of Prandtl is mentioned in [3], Reissner was not aware that Prandtl had already generalized the lateral buckling problem to include the principal bending curvature. Reissner's analysis was based on eqns (2.1) and a linearized perturbation about the deflected shape prior to buckling with the implicit assumption that $\zeta_0^2 \ll 1$. In order to make a comparison with Reissner's equation, we divide eqn (2.20) by s, differentiate with respect to s, and multiply by s^2. These operations yield a third order N_1 equation similar to Reissner's

$$s\frac{d^3N_1}{ds^3} - \frac{d^2N_1}{ds^2} + \frac{P^2}{BC}\left[s^3\left(1 - \frac{B}{A} - \frac{C}{A}\right) + \underline{s(5s^2 - l^2)\frac{C}{2A}}\right]\frac{dN_1}{ds}$$
$$+ \frac{P^2}{BC}\left[s^2\left(1 - \frac{B}{A}\right) + \underline{(3s^2 + l^2)\frac{C}{2A}}\right]N_1 = 0. \tag{3.3}$$

Equation (3.3) is identical to Reissner's eqn (5) with the exception that the two underlined terms do not appear in [3].† These underlined terms are of the same order of magnitude as the other term involving C/A. An examination of our analysis shows that these terms are a result of the $T_0\lambda$ term in the first of Kirchoff's eqns (2.1). A study of Reissner's analysis [3] reveals that he neglected this term, yielding an incomplete expression for N_1

$$N_1 = P\phi. \qquad \text{(Reissner)} \tag{3.4}$$

The complete expression for N_1 from eqn (2.14) is

$$N_1 = P\left(\phi + \int_s^l \zeta_0\lambda \, ds\right). \qquad \text{(Correct)} \tag{3.5}$$

The complete N_1 expression introduces the underlined C/A terms in eqn (3.3). This might appear to be merely a refinement of the analysis of [3] except for two important considerations. First, these terms are the same order of magnitude as the other C/A term in eqn (3.3). Second, the absence of these underlined terms prevents integration to the much simpler equation for N_1, eqn (2.20). It is interesting to note that Refs. [2] and [3] each erred in their expression for N_1. Of further interest is the fact that Reissner's expression for N_1, eqn (3.4) is inconsistent with the previously published result of Prandtl, eqn (3.2).

3.3 *Physical interpretation*

A physical interpretation of the difference between the present analysis and the analyses of [2] and [3] can be given in terms of the distinction between the geometric and elastic angles of twist. From equilibrium considerations, one can show that $N_1 = P\hat{\phi}$ where $\hat{\phi}$ is the dot product between unit vectors in the direction of N_1 and in the direction of the loading P as shown in Fig. 3. (To the order of magnitude considered here, $\sin\hat{\phi} = \hat{\phi}$). The angle $\hat{\phi}$ is the geometric angle of twist and is not, in general, equal to the elastic angle of twist ϕ. The exact relationship between these two angles is derived from purely geometric considerations in [10] and for $\zeta_0^2 \ll 1$ is given by

$$\hat{\phi} = \phi + \int_s^l \zeta_0\lambda \, ds. \tag{3.6}$$

The integral term in eqn (3.6) accounts for the fact that the N_1 axis may rotate due to the product of slope ζ_0, and lateral rotation λ ds.

The distinction between $\hat{\phi}$ and ϕ has a bearing on the N_1 equation because N_1 is directly proportional to $\hat{\phi}$. Thus eqn (3.6) and the definition of $\hat{\phi}$ yields

$$N_1 = P\hat{\phi} = P\left(\phi + \int_s^l \zeta_0\lambda \, ds\right). \tag{3.7}$$

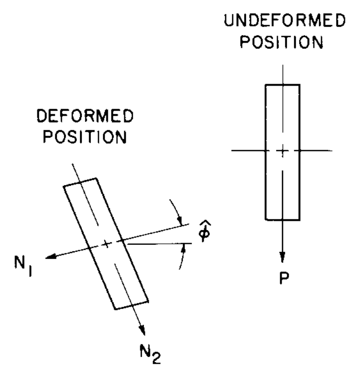

Fig. 3.

†Reissner also retained BC/A^2 with respect to unity in the $s^3(dN_1/ds)$ term as a result of the product $(1 - B/A)$ $(1 - C/A)$, although we have neglected this term in order to be consistent with $\zeta_0^2 \ll 1$.

which is equivalent to the correct equilibrium expressions, eqns (2.25) and (3.5). A review of the work of [2] and [3] shows that their analyses did not yield the correct relationship between $\hat{\phi}$ and ϕ. Although Prandtl [2] was aware that the product of bending deflections affected the geometric twist, eqn (3.2), he did not include the complete effect. Reissner, on the other hand, totally neglected the effect of bending deflections in the N_1 equilibrium eqn (3.4) and thus did not distinguish between the elastic and geometric angles of twist.

4. NUMERICAL RESULTS

4.1 *Nondimensional equations*

We will now compare the buckling load as calculated from the analysis of this paper, eqns (2.11) and (2.20), with the buckling load as calculated from eqn (35) of [2] and eqn (5) of [3]. First, we nondimensionalize the equations and transform them into a form similar to that of the classical buckling eqn (2.13). The τ equation, eqn (2.11), may be integrated once to give a buckling equation in terms of the elastic angle of twist ϕ

$$\frac{d^2\phi}{d\sigma^2}+\beta^2\left[\sigma^2\left(1-\frac{B}{A}-\frac{C}{2A}\right)\phi+\frac{C}{2A}\sigma^2\int_\sigma^1\frac{1}{\xi^2}\frac{d\phi}{d\xi}(\xi)\,d\xi\right]=0 \tag{4.1}$$

where $\sigma\equiv s/l$ and β is the nondimensional buckling load ($\beta^2\equiv P^2l^4/BC$). The N_1 equation, eqn (2.20), is already in the classical form and may be written in terms of the geometric angle of twist ($\hat{\phi}=N_1/P$)

$$\frac{d^2\hat{\phi}}{d\sigma^2}+\beta^2\left[\sigma^2\left(1-\frac{B}{A}+\frac{3C}{2A}\right)-\frac{C}{2A}\right]\hat{\phi}=0. \tag{4.2}$$

The y equation, eqn (2.26), transforms easily into eqn (4.1) and is not treated here. Equations (4.1) and (4.2) are different because they describe different physical quantities, ϕ and $\hat{\phi}$. Nevertheless, they yield the same nondimensional buckling load, β. Similarly, eqn (35) of [2] can be expressed as

$$\frac{d^2\phi}{d\sigma^2}+\beta^2\sigma^2\left\{\left(1-\frac{B}{A}\right)\phi+\frac{C}{2A}(1-\sigma^2)\left[\frac{1}{\sigma}\frac{d\phi}{d\sigma}+\int_\sigma^1\frac{1}{\xi}\frac{d^2\phi}{d\xi^2}(\xi)\,d\xi\right]\right\}=0 \tag{4.3}$$

(Prandtl)

and eqn (5) of [3] becomes

$$\frac{d^2}{d\sigma^2}\left(\frac{N_1}{P}\right)+\beta^2\left[\sigma^2\left(1-\frac{B}{A}-\frac{C}{A}\right)\frac{N_1}{P}-\frac{C}{A}\sigma\int_\sigma^1\frac{N_1}{P}(\xi)\,d\xi\right]. \tag{4.4}$$

(Reissner)

It should be noted that N_1/P in eqn (4.4) is neither ϕ nor $\hat{\phi}$ because Reissner did not distinguish between the two twist angles.

4.2 *Buckling load formulas*

The numerical values of the buckling loads as calculated from eqns (4.1)–(4.4) can be found by an asymptotic expansion in the small parameters B/A, C/A. This is a completely consistent approach since it has been assumed from the beginning that $\zeta_0^2\ll 1$ (and consequently, for B and C the same order of magnitude, B^2/A^2, C^2/A^2, $BC/A^2\ll 1$). Equations (4.1)–(4.4) may all be placed in the general form

$$\frac{d^2\theta}{d\sigma^2}+\beta^2\left(1-\frac{B}{A}\right)\sigma^2\theta-\frac{C}{A}\beta^2 f(\sigma,\theta)=0$$

$$\theta(1)=\frac{d\theta}{d\sigma}(0)=0. \tag{4.5}$$

Expansion of θ and β^2 in powers of B/A and C/A

$$\theta = \theta_0 + \frac{B}{A}\theta_1 + \frac{C}{A}\theta_2$$

$$\beta^2 = \beta_0^2 + \frac{B}{A}\beta_1^2 + \frac{C}{A}\beta_2^2 \tag{4.6}$$

and collection of like powers of C/A and B/A, yields three equations for θ_0, θ_1, and θ_2

$$\frac{d^2\theta_0}{d\sigma^2} + \beta_0^2\sigma^2\theta_0 = 0 \tag{4.7}$$

$$\frac{d^2\theta_1}{d\sigma^2} + \beta_0^2\sigma^2\theta_1 = (\beta_0^2 - \beta_1^2)\sigma^2\theta_0 \tag{4.8}$$

$$\frac{d^2\theta_2}{d\sigma^2} + \beta_0^2\sigma^2\theta_2 = \beta_0^2 f(\sigma, \theta_0) - \beta_2^2\sigma^2\theta_0. \tag{4.9}$$

Equation (4.7) has the solution

$$\theta_0 = \sqrt{\sigma} J_{-1/4}\left(\frac{\beta_0\sigma^2}{2}\right); \qquad \beta_0 = 4{\cdot}0126 \tag{4.10}$$

which is the classical deep beam buckling solution. Equation (4.8) has the trivial solution $\beta_1^2 = \beta_0^2$, $\theta_1 = 0$. Equation (4.9) can be solved by taking $\theta_2(\sigma) = \alpha(\sigma)\theta_0(\sigma)$. Thus

$$\frac{d\alpha}{d\sigma}\theta_0^2 = \beta_0^2 \int_0^\sigma \theta_0 f(\xi, \theta_0)\, d\xi - \beta_2^2 \int_0^\sigma \xi^2\theta_0^2\, d\xi. \tag{4.11}$$

The boundary condition $\theta_0(1) = 0$ immediately yields

$$\beta_2^2 = \beta_0^2 \frac{\int_0^1 \theta_0 f(\sigma, \theta_0)\, d\sigma}{\int_0^1 \sigma^2\theta_0^2\, d\sigma}. \tag{4.12}$$

Therefore, the buckling load formula can be expressed as

$$\beta^2 = \beta_0^2 \left[1 + \frac{B}{A} + \frac{C}{A}\frac{\int_0^1 \theta_0 f(\sigma, \theta_0)\, d\sigma}{\int_0^1 \sigma^2\theta_0^2\, d\sigma}\right] \tag{4.13}$$

Equation (4.13) may now be applied to eqns (4.1)–(4.4) to determine the buckling load formulas. The results for eqns (4.1) and (4.2) can be shown to be equal by the following identity obtained using eqn (4.7) and integration by parts

$$\int_0^1 \sigma^2\theta_0 \int_\sigma^1 \frac{1}{\xi^2}\frac{d\theta_0}{d\xi}\, d\xi\, d\sigma = \int_0^1 (4\sigma^2 - 1)\theta_0^2\, d\sigma. \tag{4.14}$$

The common buckling load is found to be

$$\beta = 4{\cdot}0126\left(1 + \frac{B}{2A} + 0{\cdot}6424\frac{C}{A}\right) \qquad \text{(Correct)} \tag{4.15}$$

for $\zeta_0^2 \ll 1$. The buckling load for the Prandtl equation, eqn (4.3), is

$$\beta = 4{\cdot}0126\left(1 + \frac{B}{2A} + 0{\cdot}8118\frac{C}{A}\right) \qquad \text{(Prandtl)} \tag{4.16}$$

and for the Reissner equation, eqn (4.4),

$$\beta = 4{\cdot}0126\left(1 + \frac{B}{2A} + 0{\cdot}8197\frac{C}{A}\right). \qquad \text{(Reissner)} \qquad (4.17)$$

Equation (4.16) is in agreement with the formula obtained by Prandtl [2] by series solution of his buckling equation. Reissner [3], on the other hand, did not develop an explicit buckling load formula. Instead he obtained an implicit infinite series solution

$$1 - \frac{\alpha(3\gamma-1)}{4.3.2} + \frac{\alpha^2(3\gamma-1)(7\gamma-1)}{8.7.6.4.3.2} - \frac{\alpha^3(3\gamma-1)(7\gamma-1)(11\gamma-1)}{12.11.10.8.7..6.4.3.2} + \cdots = 0 \qquad \text{(Reissner)} \qquad (4.18)$$

where $\alpha = \beta^2(1 - B/A)$ and $\gamma = 1 - C/A$. It may be shown that the solution of eqn (4.18) is in agreement with eqn (4.17) for small B/A, C/A. Although the three buckling load formulas (eqns 4.15–4.17) exhibit first order agreement for the coefficient of B/A, the coefficients of C/A in the Prandtl and Reissner formulas are in error by 26% and 28%, respectively. Federhofer [4] uses Reissner's series to generate numerical results, some of which are quoted in Timoshenko and Gere [5]. Thus, these previously published values of the buckling load do not represent accurately the first order effect of vertical bending prior to buckling.

Strictly speaking, the correct first order formula, eqn (4.15) is only valid for small values of B/A, C/A. Nevertheless, it is possible to utilize the "natural coordinates"† suggested by the structure of eqn (4.2) to improve the accuracy of the buckling load formula for larger values of B/A, C/A. Consider eqn (2.13) in which C/A is neglected with respect to unity. Although the asymptotic expansion gives the approximate formula

$$\beta = 4{\cdot}0126(1 + B/2A)$$

the exact solution is given by [1] as

$$\beta = \frac{4{\cdot}0126}{\sqrt{1 - B/A}}. \qquad \text{(Michell)} \qquad (4.19)$$

Equation (4.19) shows that the expansion term $B/2A$ in the buckling load formula is the first term of a Taylor series for the inverse square root. This suggests that an improved version of eqn (4.15) may be written in the form

$$\beta = \frac{4{\cdot}0126}{\sqrt{1 - B/A - 1{\cdot}2848C/A}} \qquad \text{(Correct, modified)} \qquad (4.20)$$

which is also the formula obtained by applying the Rayleigh–Ritz method to eqn (4.2) with θ_0 as the comparison function. Equation (4.20) has the same first order behavior as eqn (4.15) but is accurate over a larger range of B/A, C/A as shown below.

4.3 *Numerical solution of τ equation for rectangular cross section*

Equations (2.4) and (2.6) may be solved iteratively by numerical integration for comparison with the buckling load formulas, eqns (4.15), (4.16), and (4.18)–(4.20). As an illustrative example, we consider a rectangular cross section so that $B/A = b^2/h^2$ where b/h is the ratio of cross section dimensions. From [12] C/A is uniquely determined by B/A for fixed Poisson's ratio, ν.

$$\frac{C}{A} = \frac{2}{1+\nu}\frac{B}{A}\left[1 - \frac{192}{\pi^5}\sqrt{\frac{B}{A}}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^5}\tanh\left(\frac{n\pi}{2\sqrt{B/A}}\right)\right]. \qquad (4.21)$$

Results for such a rectangular cross section are shown in Figs. 4 and 5 for $\nu = 0{\cdot}3$.

†For a further discussion of "natural coordinates" see Van Dyke [11].

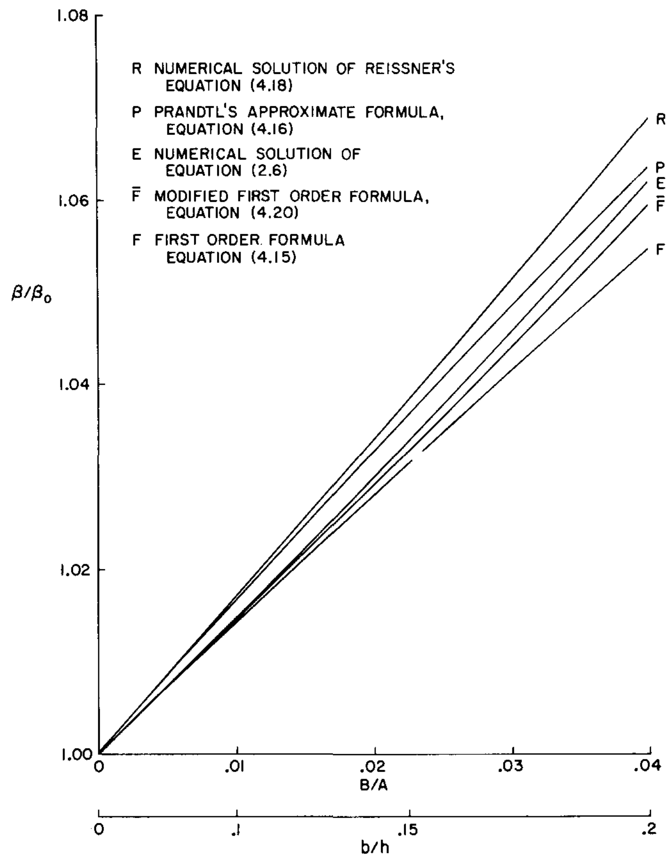

Fig. 4.

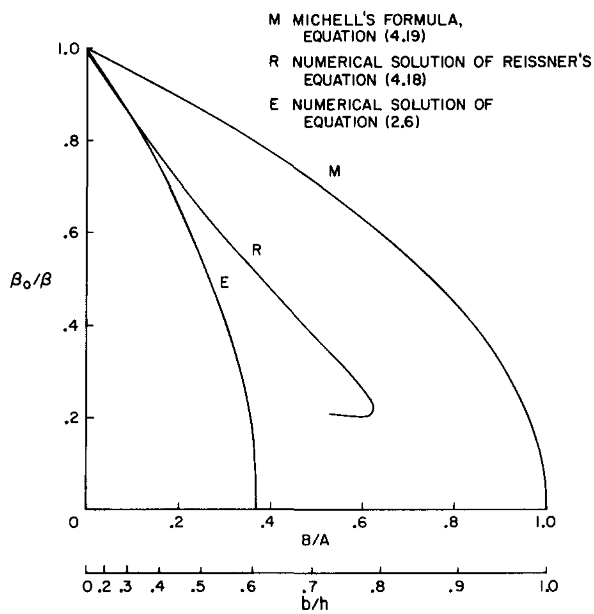

Fig. 5.

In Fig. 4 the numerical solution of the general eqn (2.6) is compared with the first order formulas, eqns (4.15) and (4.20), and with the solutions of Prandtl [2] and Reissner [3], eqns (4.16) and (4.18), respectively. The first order behavior of the solutions is exhibited by the slope of the curve at $B/A = 0$. Both of the first order formulas of this paper show the correct slope at $B/A = 0$ with the modified formula being accurate over a larger range of B/A. The formulas of

Prandtl and Reissner, on the other hand, do not predict the correct slope, each being in error by approximately 18%.

In Fig. 5, the numerical solution of the general equation is compared with the results of Reissner[3] and Michell[1], eqns (4.18) and (4.19), respectively, for larger values of B/A. Reissner's theory provides the wrong curvature with B/A and thus crosses the correct solution near $B/A = 0{\cdot}1$; this provides a temporary agreement between the two curves. This agreement vanishes, however, as the correct curve approaches an infinite buckling load at $B/A = 0{\cdot}3678$ ($b/h = 0{\cdot}6065$). The Reissner theory, on the other hand, yields the physically meaningless result of a finite buckling load at $C/A = 0{\cdot}5$ ($B/A \simeq 0{\cdot}625$) but no buckling possible for $C/A > 0{\cdot}5$. It is interesting to note that the classical Michell formula predicts an infinite buckling load at $B/A = 1$ rather than at the much lower (and less intuitive) value predicted by the general equation.

5. CONCLUSIONS

A general lateral buckling equation is derived including the complete effect of bending curvature in the plane of greatest flexural rigidity prior to buckling, eqn (2.6). This general equation is then specialized to include only the first order effect of principal bending curvature ($\zeta_0^2 \ll 1$), eqns (2.11), (2.20) and (2.26). Equation (2.20) is the most compact form of the buckling equation and is easily compared with the classical theory that neglects principal bending curvature.

Comparison of these equations with those of Prandtl[2] and Reissner[3] reveals that the previous theories are not equivalent to the present theory. The discrepancies are shown to be the result of incorrect expressions for the geometric angle of twist $\hat{\phi}$ in [2, 3]. Prandtl did not include the correct effect of bending on $\hat{\phi}$, and Reissner did not include the effect at all. Curiously enough, inclusion of the proper terms actually simplifies the buckling equations.

The specialized equations are used to generate a first order buckling load formula, eqn (4.15), which is noticeably different from similar formulas derived from the equations of [2, 3]. The numerical results of Federhofer[4], some of which are quoted by Timoshenko and Gere[5], were calculated using the equation of [3] and thus are not accurate. A modified formula is introduced, eqn (4.20), which extends the validity of the specialized theory to larger values of B/A and C/A. Both present first order formulas compare favorably with the numerical solution of the general eqn (2.6) for the case of a rectangular cross section, while results from previously published theories do not show agreement with the numerical solution, Figs. 4 and 5. This numerical solution is valid for all $B/A \leq 1$ and predicts an infinite buckling load at $B/A = 0{\cdot}3678$.

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